Séance du 08/10/19.

Suites numériques.

Rappel :

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$
$$x \longmapsto f(x).$$ (fonction).

$$\mathbb{N}$$
$$n \longmapsto u_n$$

$\mathbb{N}$: $\{0;1;2$ etc... $\}$.

indice → n	0	1	2	3	4	5	...
terme → u_n	u_0	u_1	u_2	u_3	u_4	u_5	
	1^{er} terme	2^e	3^e	4^e	5^e	6^e	

u_4 : $5^{ème}$ terme.

u_8 : $9^{ème}$

u_n : $(n+1)^{ème}$ terme.

$$n \xmapsto{(u_n)} u_n$$

Application 1 :

$U_n = \frac{1}{n^2+4}$.

$U_0 = \frac{1}{0^2+4} = \frac{1}{4}$

$U_2 = \frac{1}{2^2+4} = \frac{1}{8}$.

$U_1 = \frac{1}{1^2+4} = \frac{1}{5}$.

$\begin{cases} U_0 = 1 \\ U_{n+1} = 2 \times U_n \end{cases}$.

$U_1 = U_{0+1} = 2 \times U_0$

$= 2 \times 1 = 2$.

Application 2.

$\begin{cases} U_0 = 0 \\ U_{n+1} = U_n + 2n + 1 \end{cases}$.

$U_0 = 0$.

$U_1 = U_0 + 2 \times 0 + 1 = 0 + 0 + 1 = 1$.

$U_2 = U_1 + 2 \times 1 + 1 = 1 + 2 + 1 = 4$

$U_3 = U_2 + 2 \times 2 + 1 = 4 + 4 + 1 = 9$

$U_4 = U_3 + 2 \times 3 + 1 = 9 + 6 + 1 = 16$.

$* \ (a+b)^2 = a^2 + 2ab + b^2$.

Application 3.

$n \in \mathbb{N} \quad U_n = n^2$.

$U_4 = 4^2$

$U_6 = 6^2$

$U_{abc} = (abc)^2$

$U_{n+1} - U_n = (n+1)^{2} - n^2$ *

$= n^2 + 2n + 1 - n^2$

$= 2n+1.$

Or

$n \geq 0$ car $n \in \mathbb{N}.$

$2n \geq 2 \times 0$

$2n \geq 0$

$2n+1 \geq 1 > 0.$

donc $2n+1 > 0$

$u_{n+1} - u_n > 0$ donc (u_n) est strictement croissante.

$$\frac{u_{n+1}}{u_n} > 1$$

$\times u_n > 0$

$$\cancel{u_n} \times \frac{u_{n+1}}{\cancel{u_n}} > 1 \times u_n.$$

$$u_{n+1} > u_n.$$

<u>Application 4.</u> Démontrons que $\forall n \in \mathbb{N}$, $u_n = 2^n$ est une suite croissante

$u_n = \underbrace{2 \times 2 \times 2 \times \ldots \times 2}$ > 0

.

n fois

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{2^n} = \frac{\cancel{2^n} \times 2}{\cancel{2^n}}$$

$a^b \times a^c = a^{b+c}$

donc $\boxed{2^n \times 2^1 = 2^{n+1}}$

$$= 2 > 1$$

donc (u_n) est croissante.

Exercice 1

Pour les suites suivantes, trouver la fonction f associée à la suite définie par la relation de récurrence $u_{n+1} = f(u_n)$ et calculer les termes de u_1 à u_4

a) $\begin{cases} u_0 = 5 \\ u_{n+1} = \dfrac{2u_n}{u_n + 1} \end{cases}$ b) $\begin{cases} u_0 = -1 \\ u_{n+1} = (u_n + 1)^2 \end{cases}$ c) $\begin{cases} u_0 = 2 \\ u_{n+1} = \dfrac{u_n - 1}{u_n} \end{cases}$ d) $\begin{cases} u_0 = 1 \\ u_{n+1} = \sqrt{u_n + 1} \end{cases}$

a) $\begin{cases} u_0 = 5 \\ u_{n+1} = \dfrac{2 \times u_n}{u_n + 1} \end{cases}$

$$u_1 = \frac{2 \times u_0}{u_0 + 1} = \frac{2 \times 5}{5+1} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

$$u_2 = \frac{2 \times u_1}{u_1 + 1} = \frac{2 \times \frac{5}{3}}{\frac{5}{3} + 1} = \frac{\frac{10}{3}}{\frac{5}{3} + \frac{3}{3}} = \frac{\frac{10}{3}}{\frac{8}{3}} = \frac{10}{\cancel{3}} \times \frac{\cancel{3}}{8}$$

$$= \frac{10}{8} = \frac{5}{4}.$$

$$u_3 = \frac{2 \times u_2}{u_2 + 1} = \frac{2 \times \frac{5}{4}}{\frac{5}{4} + 1} = \frac{\frac{10}{4}}{\frac{5}{4} + \frac{4}{4}} = \frac{\frac{10}{4}}{\frac{9}{4}} = \frac{10}{\cancel{4}} \times \frac{\cancel{4}}{9} = \frac{10}{9}.$$

$$u_4 = \frac{2 \times u_3}{u_3 + 1} = \frac{2 \times \frac{10}{9}}{\frac{10}{9} + 1} = \frac{\frac{20}{9}}{\frac{10}{9} + \frac{9}{9}} = \frac{\frac{20}{9}}{\frac{19}{9}} = \frac{20}{\cancel{9}} \times \frac{\cancel{9}}{19} = \frac{20}{19}.$$

b) $\begin{cases} u_0 = -1 \\ u_{n+1} = (u_n + 1)^2 \end{cases}$

$$u_1 = (u_0+1)^2 = (-1+1)^2 = 0^2 = 0.$$

$$u_2 = (u_1+1)^2 = (0+1)^2 = 1^2 = 1.$$

$$u_3 = (u_2+1)^2 = (1+1)^2 = 2^2 = 4.$$

$$u_4 = (u_3+1)^2 = (4+1)^2 = 5^2 = 25.$$

c) $\begin{cases} u_0 = 2 \\ u_{n+1} = \dfrac{u_n - 1}{u_n} \end{cases}$

$$u_1 = \frac{u_0 - 1}{u_0} = \frac{2-1}{2} = \frac{1}{2}.$$

$$u_2 = \frac{u_1 - 1}{u_1} = \frac{\frac{1}{2} - 1}{\frac{1}{2}} = \frac{\frac{1}{2} - \frac{2}{2}}{\frac{1}{2}}$$

$$u_2 = \frac{-\frac{1}{2}}{\frac{1}{2}} = -\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{1} = -1.$$

$$u_3 = \frac{u_2 - 1}{u_2} = \frac{-1-1}{-1} = \frac{-2}{-1} = 2.$$

$$u_4 = \frac{u_3 - 1}{u_3} = \frac{2-1}{2} = \frac{1}{2}.$$

d) $\begin{cases} u_0 = 1 \\ u_{n+1} = \sqrt{u_n + 1} \end{cases}$

$$u_1 = \sqrt{u_0 + 1} = \sqrt{1+1} = \sqrt{2}.$$

$$u_2 = \sqrt{u_1 + 1} = \sqrt{\sqrt{2} + 1}.$$

$$u_3 = \sqrt{u_2 + 1} = \sqrt{\sqrt{\sqrt{2}+1} + 1}$$

$$u_4 = \sqrt{u_3 + 1} = \sqrt{\sqrt{\sqrt{\sqrt{2}+1}+1}+1}.$$

Exercice 2

Pour les suites suivantes, calculer les termes de u_1 à u_5 puis **conjecturer une formule explicite** du terme général. Retrouver alors u_0 à partir de la formule conjecturée puis démontrer la relation donnée entre u_{n+1} et u_n.

a) $\begin{cases} u_0 = 1 \\ u_{n+1} = \dfrac{1}{2}u_n \end{cases}$ b) $\begin{cases} u_0 = 1 \\ u_{n+1} = u_n + 5 \end{cases}$ c) $\begin{cases} u_0 = 1 \\ u_{n+1} = 1 - \dfrac{1}{1+u_n} \end{cases}$

a) $\begin{cases} u_0 = 1 \\ u_{n+1} = \frac{1}{2} u_n \end{cases}$

$u_1 = \frac{1}{2} \times u_0 = \frac{1}{2} \times 1 = \frac{1}{2}.$

$u_2 = \frac{1}{2} \times u_1 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = \left(\frac{1}{2}\right)^2.$

$u_3 = \frac{1}{2} \times u_2 = \frac{1}{2} \times \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^3.$

$u_4 = \frac{1}{2} \times u_3 = \frac{1}{2} \times \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^4$

$u_5 = \frac{1}{2} \times u_4 = \frac{1}{2} \times \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^5.$

Conjecture $u_n = \left(\frac{1}{2}\right)^n.$

$u_{n+1} = \left(\frac{1}{2}\right)^{n+1} = \left(\frac{1}{2}\right)^n \times \left(\frac{1}{2}\right)$

$u_{n+1} = u_n \times \frac{1}{2} = \frac{1}{2} \times u_n.$

b) $\begin{cases} u_0 = 1 \\ u_{n+1} = u_n + 5. \end{cases}$

$u_1 = u_0 + 5 = 1 + 5 = 6.$

$u_2 = u_1 + 5 = 6 + 5 = 11.$

$u_3 = u_2 + 5 = 11 + 5 = 16.$

$$u_4 = u_3 + 5 = 16 + 5 = 21.$$

$$u_5 = u_4 + 5 = 21 + 5 = 26.$$

$$u_n = 5n + 1.$$

$$u_0 = 5 \times 0 + 1 = 1 \qquad u_5 = 5 \times 5 + 1 = 26.$$

$$u_{n+1} = 5(n+1) + 1$$

$$u_{n+1} = 5n + 5 + 1$$

$$u_{n+1} = 5n + 1 + 5.$$

$$\boxed{u_{n+1} = u_n + 5.}$$

c) $\begin{cases} u_{n+1} = 1 - \dfrac{1}{1+u_n} \\ u_0 = \dfrac{1}{1} \end{cases}$

$$u_1 = 1 - \frac{1}{1+u_0}$$

$$u_1 = 1 - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$u_2 = 1 - \frac{1}{1+u_1} = 1 - \frac{1}{1+\frac{1}{2}} = 1 - \frac{1}{\frac{3}{2}}$$

$$= 1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}.$$

$$u_3 = 1 - \frac{1}{1+u_2} = 1 - \frac{1}{1+\frac{1}{3}} = 1 - \frac{1}{\frac{4}{3}} = 1 - \frac{3}{4} = \frac{1}{4}.$$

$$u_4 = 1 - \frac{1}{1+u_3} = 1 - \frac{1}{1+\frac{1}{4}} = 1 - \frac{1}{\frac{5}{4}} = 1 - \frac{4}{5} = \frac{1}{5}.$$

Conjecture: $\boxed{u_n = \frac{1}{n+1}}$ $\qquad u_3 = \frac{1}{4}$.

$$u_{n+1} = \frac{1}{n+1+1} = \frac{1}{n+2}. \qquad u_{n+1} = 1 - \frac{1}{1+u_n}.$$

Or $1 - \frac{1}{1+u_n} = 1 - \frac{1}{1 + \frac{1}{n+1}}$

$$= 1 - \frac{1}{\frac{n+1}{n+1} + \frac{1}{n+1}}$$

$$= 1 - \frac{1}{\frac{n+2}{n+1}}$$

$$= 1 - \frac{n+1}{n+2}$$

$$= \frac{n+2}{n+2} - \frac{n+1}{n+2} = \boxed{\frac{1}{n+2}}.$$